



TITLE:

# A transversality condition for quadratic family at Collet-Eckmann parameter (Problems on complex dynamical systems)

AUTHOR(S):

Tsujii, Masato

---

CITATION:

Tsujii, Masato. A transversality condition for quadratic family at Collet-Eckmann parameter (Problems on complex dynamical systems). 数理解析研究所講究録 1998, 1042: 99-101

ISSUE DATE:

1998-04

URL:

<http://hdl.handle.net/2433/62094>

RIGHT:

# A transversality condition for quadratic family at Collet-Eckmann parameter

Masato Tsujii (Hokkaido University)

January 6, 1998

We consider real quadratic maps  $Q_t : \mathbf{R} \rightarrow \mathbf{R}$ ,  $x \mapsto t - x^2$ , where  $t \in \mathbf{R}$  is a parameter. We say that  $Q_t$  satisfies Collet-Eckmann condition if

$$\liminf_{n \rightarrow \infty} \sqrt[n]{|DQ_t^n(Q_t(0))|} > 1.$$

This condition implies that the dynamics of  $Q_t$  is 'chaotic' (existence of absolutely continuous invariant measure, decay of correlation, etc.). We give

**Theorem 1** *If  $Q_t$  satisfies Collet-Eckmann condition, then*

$$\lim_{n \rightarrow \infty} \frac{\frac{\partial}{\partial s} \{Q_s^n(0)\}|_{s=t}}{DQ_t^{n-1}(Q_t(0))} > 0. \quad (1)$$

In a sense, the condition(1) implies that the quadratic family is transversal to the "manifold" of the maps which is topologically conjugate to  $Q_t$ .

Combining theorem 1 with Jacobson's theorem [2], we get

**Proposition 2** *Let  $A$  be the set of parameters  $t$  for which  $Q_t$  satisfies Collet-Eckmann condition and*

$$\liminf_{n \rightarrow \infty} n^{-1} \log |DQ_t(Q_t^n(0))| = 0. \quad (2)$$

*Then every point in  $A$  is a density point of  $A$  itself in the interval  $[0, 2]$ .*

Remark that  $A$  contains  $t = 2$ . The condition (2) holds if the critical point 0 is not recurrent.

We prove theorem 1 as follows. Take  $r > 1$  such that

$$\liminf_{n \rightarrow \infty} \sqrt[n]{|DQ_t^n(Q_t(0))|} > r > 1.$$

We consider  $Q_t$  as a map from the complex plane to itself. Let  $A$  be a Ruelle operator  $A$  on the quadratic differentials:

$$A(\varphi)(x) = \sum_{Q_t(y)=x} \frac{\varphi(y)}{[DQ_t(y)]^2},$$

acting on the space

$$S = \left\{ x = \sum_{i=1}^{\infty} x_i \psi_i \mid \sum_i |x_i DQ^i(Q(0))| r^{-i} < \infty \right\}$$

where  $\psi_i(z) = (z - Q^i(0))^{-1}$ . We endow  $S$  with a norm

$$|x| = \sum_i |x_i DQ^i(Q(0))| r^{-i}.$$

Then we have, formally,

$$\lim_{n \rightarrow \infty} \frac{\frac{\partial}{\partial s} Q_s^n(0)|_{s=t}}{DQ_t^{n-1}(Q_t(0))} = \det(\text{Id} - A). \quad (3)$$

Comparing  $A$  with the Perron-Frobenius operator, we see that the spectral radius of  $A$  is smaller than 1. Hence, if  $A$  were a finite-dimensional operator, these would imply (1). Actually, we can't give any appropriate definition for the determinant in (3) since  $A$  is an infinite dimensional operator. Instead, we approximate  $A$  by a sequence of finite-dimensional operators. For detail, see [3].

## References

- [1] M.Dunford, J.T.Schwartz, Linear operators 1, Interscience, New York,(1958)

- [2] M. Tsujii, Positive Lyapunov exponent in families of one dimensional dynamical systems, *Invent. math.* vol.111 (1993), 113–137
- [3] M. Tsujii, A simple proof for monotonicity of entropy in the quadratic family, preprint, Hokkaido University